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Weir flows and waterfalls

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Two-dimensional free-surface flows, which are uniform far upstream in a channel of finite depth that ends suddenly, are computed numerically. The ending is in the form of a vertical wall, which may force the flow upward before it falls down forever as a jet under the effect of gravity. Both subcritical and supercritical solutions are presented. The subcritical solutions are a one-parameter family of solutions, the single parameter being the ratio between the height of the wall and the height of the uniform flow far upstream. On the other hand, the supercritical solutions are a two-parameter family of solutions, the second parameter being the Froude number. Moreover, for some combinations of the parameters, it is shown that the solution is not unique.

1. Introduction

Free-surface flows past a bluff obstacle in the presence of gravity are difficult to compute. Even for the simplest geometries in steady flow, analytical solutions are rare (see Wehausen & Laitone 1960 for a few examples) and solutions must be computed numerically. But numerical solutions are also difficult to obtain, and accurate solutions to problems with a relatively simple geometrical configuration have been obtained only recently or in many cases have not been achieved yet. The difficulty of these problems comes from the presence of one or more free surfaces, whose shape must be found as part of the solution and on which the dynamic boundary condition to be satisfied is highly nonlinear.

Here, two-dimensional free-surface flows having two free surfaces forming a jet which falls down forever under the effect of gravity are considered. More precisely, we consider a flow that is uniform far upstream in a channel of finite depth that ends suddenly. The ending can be in the form of a simple sharp-edged bottomless chasm, into which the stream falls forever. More generally, the stream may be forced temporarily upward by a vertical wall at the end of the channel, but in any case, it subsequently again falls down forever as a jet. The velocity and the height of the uniform flow far upstream are denoted by U and H respectively. The Froude number upstream is defined by

$$F = \frac{U}{(gH)^{\frac{1}{2}}},$$

where g is the acceleration due to gravity. In the remainder of the paper, flows for

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which $F < 1$ will be referred to as weir flows and flows for which $F > 1$ as waterfall flows (even if the wall height is non-zero). A weir flow is shown in figure 1(*a*) and a waterfall flow is shown in figure 13.

The interesting characteristic of weir flows is that one cannot specify both the Froude number upstream *and* the height of the weir. In other words, the velocity U and the height H of the flow far upstream adjust themselves to the height W of the weir, and hence determine the Froude number. We may consider that F is a function of the dimensionless weir height W/H. For subcritical flows, there is therefore a one-parameter family of solutions, the single parameter being W/H.

The main result of this paper is to show the existence of a two-parameter family of solutions for waterfall flows, the parameters being F and W/H.

In this paper, some open questions about weir flows and waterfall flows will be answered but unfortunately, a number of questions will remain unanswered. Here are some questions of interest:

(i) Thin-weir solutions have been computed accurately by Vanden-Broeck & Keller (1987*b*) for small Froude numbers. Since the primary aim of their work was to compare their numerical results with experimental data, which are only available for small Froude numbers, they did not consider Froude numbers larger than 0.3. Can thin-weir solutions be obtained for any Froude number smaller than 1, or do they break down before reaching $F = 1$?

(ii) Waterfall solutions with $W = 0$ have been computed accurately by Vanden-Broeck (unpublished computations) for Froude numbers larger than 1. Can such solutions be obtained for Froude numbers close to but smaller than 1? Results by Chow & Han (1979) and Smith & Abd-el-Malek (1983) seem to suggest that the answer is yes.

(iii) Supercritical solutions with $W \neq 0$ have been conjectured by Vanden-Broeck & Keller (1987*b*) but not computed yet. For a given Froude number, how high can the wall be before the solution breaks down? In the limit as the velocity at the highest point on the free surface approaches zero, is there a solution with a stagnation point with a 120° angle? Is it possible to predict whether or not the upper free surface will go up first before falling?

(iv) If the wall is inclined to the left when the stream is flowing from left to right or if the wall is too high, is it possible to obtain solutions in which the jet first goes up, then curves back around to the left and finally falls down, splashing on top of the incident flow? This is a question of importance in ship hydrodynamics, see for example Tuck & Vanden-Broeck (1985). Most ships create a bow splash, and the initial motivation for our work was to model this splash by a jet rising along the bow of the ship and then falling back upon the oncoming stream. This objective turned out to be too difficult to reach, and is left for further work (see Tuck 1990 and Dias & Christodoulides 1991).

The paper is organized as follows. In §2, the general problem is formulated. Section 3 deals with subcritical flows (weir flows). In §4, we consider the special case when the water depth is infinite but we allow the wall to be inclined to the vertical. In §5, we consider the special case when the height of the wall is zero (waterfall with $W = 0$). In §6, supercritical solutions are presented. The limiting configurations with zero gravity are presented in §7 because that problem can be solved analytically.

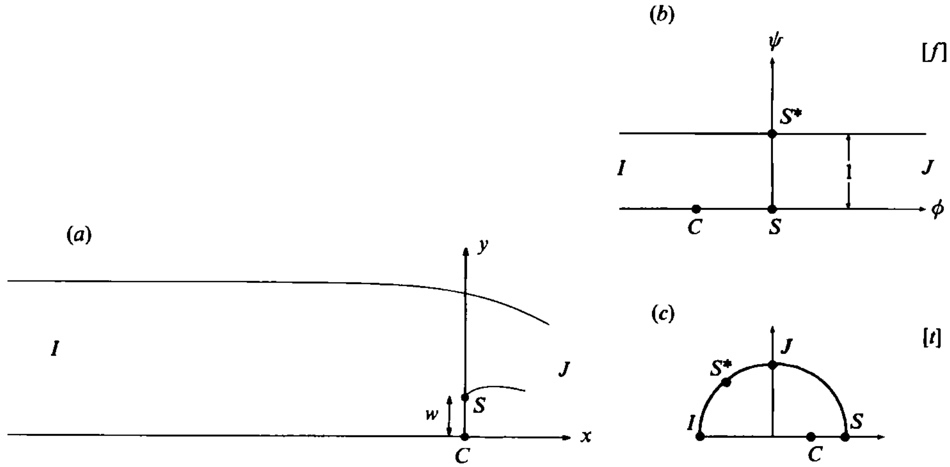

FIGURE 1. (*a*) Liquid flowing over a thin weir. The flow shown here was calculated for $F = 0.5$. S is the detachment point. The dimensionless elevation of the free surface far upstream is 1. The resulting t_c is 0.12 and $w = 0.25$. The horizontal scale is the same as the vertical scale. (*b*) Plane of the complex potential. The dimensionless height of the strip is 1. (*c*) Intermediate plane ([t]-plane).

2. Formulation of the problem

The formulation of the problem follows closely that of Vanden-Broeck & Keller (1987*b*). The steady irrotational flow of an incompressible inviscid fluid past a vertical wall is considered (see figure 1*a*). The x-axis is chosen to be along the bottom of the channel. The y-axis is chosen to be up the vertical wall, so that the origin lies at the corner C between the channel and the wall. The flow is assumed to be uniform far upstream (point I). Past the wall, the flow becomes a jet which falls down to infinity (point J). The top of the vertical wall, where the flow detaches, is denoted by S. There are two free surfaces: the upper free surface that extends from I to J and the lower free surface that extends from the detachment point S to J.

The problem is non-dimensionalized by taking U as the unit velocity and H as the unit length. The dimensionless volume discharge is therefore unity. From now on, all the new variables will be dimensionless and will be described by lower-case characters. For example, w will represent the dimensionless height W/H of the weir, x and y will be the dimensionless coordinates. In dimensionless variables, Bernoulli's equation in the entire fluid takes the form

$$\tfrac{1}{2}(u^2+v^2)+y/F^2+p = \tfrac{1}{2}+1/F^2,$$

where u and v are the horizontal and vertical components of velocity, p the pressure.

We denote the velocity potential by $\phi(x,y)$ and the stream function by $\psi(x,y)$. In addition we introduce the complex variables $z = x+\mathrm{i}y$ and $f = \phi+\mathrm{i}\psi$. The flow domain on the [f]-plane is shown in figure 1(*b*). It is an infinite strip of height 1. The left part of the bottom of the strip ($\psi = 0, \phi < 0$) represents the bottom of the channel and the wall. The right part of the bottom of the strip ($\psi = 0, \phi > 0$) represents the lower free surface, and the top of the strip is the image of the upper free surface. The origin of the complex potential f is chosen at the detachment point S. The point S^*, which lies on the upper free surface and will be used below, is the point of coordinates ($\phi = 0, \psi = 1$) in the [f]-plane.

We introduce the hodograph variable

$$\zeta(z) = \mathrm{d}f/\mathrm{d}z(z) = u-\mathrm{i}v.$$

The kinematic boundary conditions on the solid boundaries can be written as

$$\left.\begin{aligned} \operatorname{Im}\zeta = 0 \quad &\text{on} \quad \psi = 0, \phi < \phi(C), \\ \operatorname{Re}\zeta = 0 \quad &\text{on} \quad \psi = 0, \phi(C) < \phi < 0, \end{aligned}\right\} \tag{1}$$

and the dynamic boundary conditions on the free surfaces as

$$\frac{|\zeta|^2}{2} + \frac{y}{F^2} = \frac{1}{2} + \frac{1}{F^2} \quad \text{on} \quad \psi = 1 \quad \text{and} \quad \psi = 0, \phi > 0. \tag{2}$$

The domain occupied by the fluid in the $[f]$-plane is transformed into the upper half of the unit disk in the $[t]$-plane so that the points I, S and J are mapped into the points $-1, 1$ and i (see figure 1c). The bottom of the channel and the wall go onto $[-1, 1]$, the lower free surface onto the upper right quarter of the unit circle and the upper free surface onto the upper left quarter of the unit circle. We denote the image of the point C by t_c. One can think of t_c as the 'height' of the weir in the $[t]$-plane. The image of the point S^* is denoted by t^*. The transformation from the $[f]$-plane to the $[t]$-plane can be written in differential form as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{2}{\pi}\left[\frac{1-t}{(t+1)(t^2+1)}\right] \tag{3}$$

or in integrated form as

$$f = \frac{1}{\pi}\ln\left[\frac{(t+1)^2}{2(t^2+1)}\right]. \tag{4}$$

It is easy to show that

$$t^* = \mathrm{e}^{\mathrm{i}\cos^{-1}(-\frac{1}{3})}.$$

The problem to be solved is to find ζ as an analytic function of t satisfying the kinematic boundary conditions (1) on the real diameter $t \in [-1, 1]$ and the dynamic boundary conditions (2) on the free surfaces. Points on the free surfaces are represented by $t = \mathrm{e}^{\mathrm{i}\sigma}$, where $\sigma \in [0, \frac{1}{2}\pi[$ on the lower portion and $\sigma \in]\frac{1}{2}\pi, \pi]$ on the upper portion.

Before looking for possible solutions, it is of interest to determine the stagnation level, that is to say the maximum elevation y_{stag} that the free surface can ever reach. It will be reached when there is a stagnation point on the upper free surface. From Bernoulli's equation (2), one obtains

$$y_{\text{stag}} = 1 + \tfrac{1}{2}F^2. \tag{5}$$

3. Weir flows

In this section, we assume that F is smaller than 1, that is to say that the flow is subcritical upstream. This is the thin-weir problem in finite depth, which has been studied by various investigators. The solution presented below is similar to Vanden-Broeck & Keller's solution (1987b). The motivation however is different. Vanden-Broeck & Keller were interested in showing how their numerical results compared with experimental results. For that reason, they computed solutions with small Froude numbers (typically less than 0.3). The purpose here is to obtain an overall view of the problem and to determine for what range of F solutions exist. Downstream, as $\phi \to +\infty$, the flow becomes a jet and ζ increases like $f^{\frac{1}{3}}$. Around the corner C, ζ vanishes like $(t-t_c)^{\frac{1}{2}}$. Therefore we write the hodograph variable ζ as

$$\zeta = -\mathrm{i}(t-t_c)^{\frac{1}{2}}[-\ln c(1+t^2)]^{\frac{1}{3}}\mathrm{e}^{\Omega(t)}, \tag{6}$$

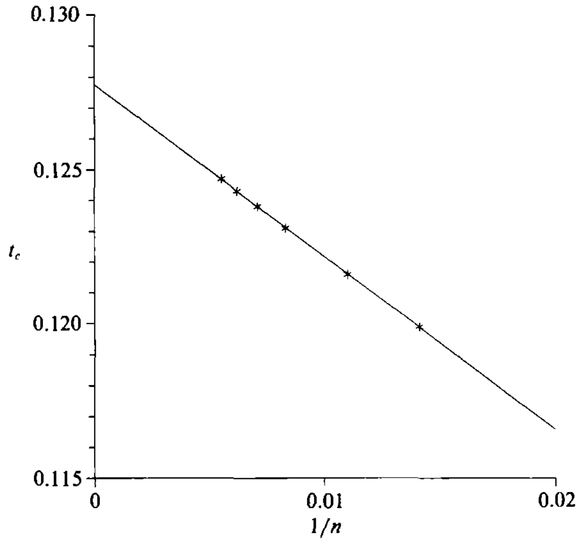

FIGURE 2. Output t_c for subcritical weir flows as a function of the number n of collocation points into which the free surface is discretized. The Froude number is 0.5.

where c is a constant chosen between 0 and $\frac{1}{2}$. The only role of the constant c is to prevent the argument of the logarithm from being equal to 1 when $t = 0$. Otherwise, the velocity would be zero at the point $t = 0$. The function $\Omega(t)$ is analytic inside the unit disk and continuous on the unit circle and can therefore be expanded in a power series

$$\Omega(t) = \sum_{m=0}^{\infty} a_m t^m. \tag{7}$$

The coefficients a_m, which must be real in order to satisfy the kinematic boundary conditions (1) on $t \in [-1, 1]$, are determined by collocation. To do so we truncate the infinite series in (7) after $n-1$ terms and we introduce $n-1$ mesh points on the free surface $t_l = e^{i\sigma_l}$, where

$$\sigma_l = \frac{\pi}{n-1}(l-\tfrac{1}{2}), \quad l = 1, \ldots, n-1. \tag{8}$$

These points are equally spaced on the upper half of the unit circle and there is the same number of points on both free surfaces. By using (6) we obtain the values of u and v at the points σ_l in terms of the coefficients a_m. The y-values are obtained by integrating

$$dz = \frac{1}{\zeta} df = \frac{1}{\zeta} \frac{df}{dt} dt \tag{9}$$

along the free surface, yielding

$$x + iy = iw + \frac{1}{\pi} \int_0^{\sigma} \left(\frac{\sin \frac{1}{2}\sigma}{\cos \frac{1}{2}\sigma \cos \sigma} \right) \left[\operatorname{Re}\left(\frac{1}{\zeta}\right) + i \operatorname{Im}\left(\frac{1}{\zeta}\right) \right] d\sigma. \tag{10}$$

Substituting the expressions for u, v and y into (2) at the points σ_l, we obtain $n-1$ nonlinear algebraic equations for the n unknowns t_c and a_m $(0 \leqslant m \leqslant n-2)$. The nth equation is obtained by requiring the velocity to be 1 far upstream: $\zeta(-1) = 1$. The only parameter which is specified is the Froude number.

FIGURE 3. Profile of the free surfaces for $F = 1$. The resulting t_c is 0.48 and $w = 0.09$. The horizontal scale is the same as the vertical scale.

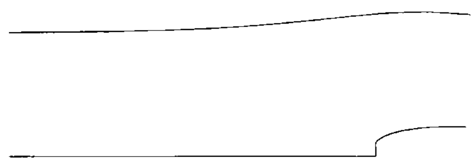

FIGURE 4. Same as figure 3 but for $F = 1.6$. $t_c = 0.48$ and $w = 0.11$.

This system of n nonlinear equations with n unknowns is solved by Newton's method for given values of F. The C05NBF package of the NAG library was used for solving the system. The error was found to decay like $1/n$. This is confirmed by figure 2, which shows that the final output of t_c is a linear function of $1/n$. This n^{-1} convergence is typical of the series truncation method that has also been used successfully by Dias, Keller & Vanden-Broeck (1988) for a variety of free-surface problems.

The program converged for all values of F between 0 and 1, and we can conclude that a thin-weir solution exists for every Froude number between 0 and 1. For small values of F, excellent agreement was found with the solutions of Vanden-Broeck & Keller (1987*b*). For values of F close to 1, more coefficients were needed in the power series for $\Omega(t)$. Typical profiles for $F = 0.5$ and $F = 1.0$ are shown in figures 1(*a*) and 3. In the profile for $F = 1.0$, up to 180 coefficients were used in the series. The upper free surface was found to go up very slightly before going down. The maximum value for y on the upper free surface was 1.0016.

Since the program was still converging for $F = 1$, we decided to test values of the Froude number larger than 1, and convergence was obtained up to $F = 1.6$. Figure 4 shows the profile of the free surfaces for $F = 1.6$. In figure 5, a plot of $w = W/H$, the dimensionless height of the weir, versus F is shown. This shows that w decreases down to about 0.07 before increasing again. The value of the Froude number corresponding to $w_{\min}$ is about 1.3. The fact that the branch of subcritical solutions does not stop at $F = 1$ has already been encountered by Vanden-Broeck & Keller (1987*a*) in their study of free-surface flows due to a sink. In §6, we will show that there are additional supercritical solutions.

The subcritical solutions described above are characterized by a uniform stream far upstream. These are therefore special solutions, since subcritical solutions usually have a train of waves at infinity. We conjecture that subcritical solutions with waves also exist. The existence of wave-free solutions is similar to the existence of conjugate flows in the presence of an obstacle in an open channel (Dias & Vanden-Broeck 1989). Conjugate flows, which are subcritical upstream and supercritical downstream of the obstacle, also are a one-parameter family of solutions. In other words, for a given height of the obstacle, there is only one possible value of the upstream Froude number which gives a wave-free solution.

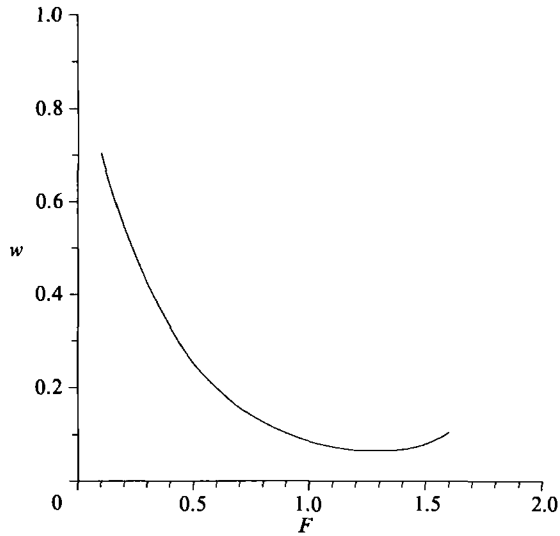

FIGURE 5. Plot of the dimensionless height of the weir w as a function of the Froude number.

4. Inclined weir in water of infinite depth

Before reporting on the supercritical solutions, we present results for a thin weir in water of infinite depth when the weir is inclined to the vertical (see figure 6*a*). This problem was recently solved by Vanden-Broeck & Keller (1986) for all values of the angle α between the weir and the horizontal. For this problem, a different non-dimensionalization must be used. Let $(Qg)^{\frac{1}{3}}$ be the unit velocity and $(Q^2/g)^{\frac{1}{3}}$ be the unit length, where Q is the flux. Let h_s denote the dimensionless elevation of the free surface at $-\infty$ above the top of the weir (point S). In this section, we study the variation of h_s as a function of α. The origin of the coordinate system is located at the detachment point S. The $[t]$-plane and the $[f]$-plane are identical to the planes shown in figures 1(*b*) and 1(*c*). However, the expression for ζ is different because there is now a source-type singularity at point I and the corner C has been removed. The new expression for ζ is

$$\zeta = \mathrm{e}^{-\mathrm{i}\alpha}(1+t)^{2\alpha/\pi}[-\ln c(1+t^2)]^{\frac{1}{3}}\,\mathrm{e}^{\Omega(t)}. \tag{11}$$

The same series truncation procedure is used. Again, $n-1$ terms are retained in the series. There is another unknown: h_s. The choice of the collocation points is the same as before. However, the nth equation is obtained by requiring that Bernoulli's constant be the same on the lower and upper free surfaces. It can be done by computing in two different ways the elevation of the point S^* whose coordinates in the $[f]$-plane are $(\phi = 0, \psi = 1)$. There is no particular reason for choosing that point except that it is the point on the upper free surface that lies on the same equipotential as the detachment point. The first way is to use the expression for y given above (equation (10)). The second way is to integrate $\mathrm{d}z = \zeta^{-1}\,\mathrm{d}f$ along the equipotential $\phi = 0$. We obtain

$$y(S^*) = \int_0^1 \frac{1}{\zeta}\,\mathrm{d}\psi. \tag{12}$$

In (11), ζ is given in terms of t but ζ can be expressed in terms of ψ by inverting (4).

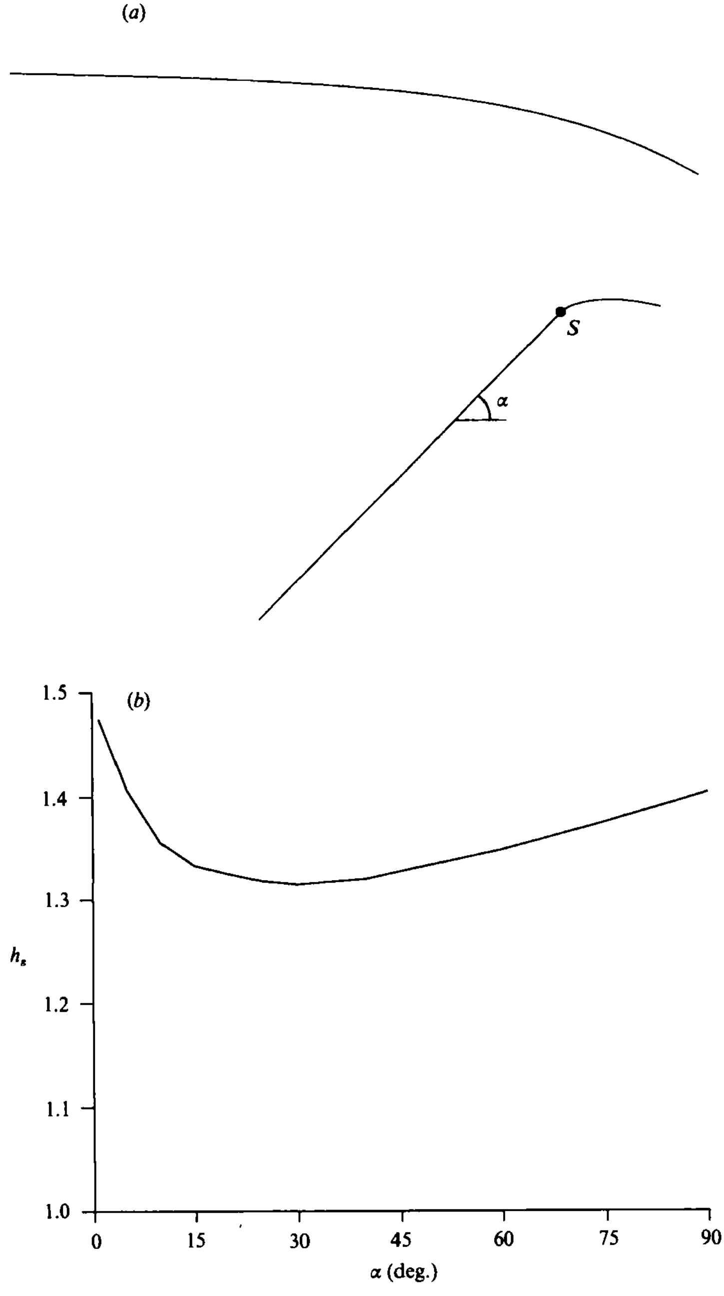

FIGURE 6. (a) Liquid flowing over an inclined weir. Computed solution for $\alpha = 45°$. (b) Plot of the elevation of the free surface relative to the elevation of the top of the weir as a function of the angle of inclination of the weir.

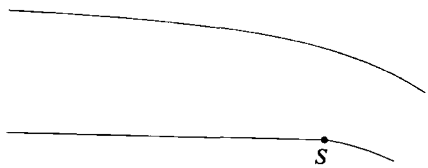

FIGURE 7. Computed waterfall for $F = 1$ and $n = 180$. The horizontal scale is the same as the vertical scale.

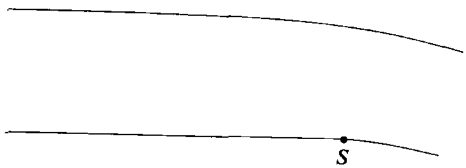

FIGURE 8. Same as figure 7 but for $F = 2$ and $n = 140$.

Figure 6(*a*) shows a computed solution for $\alpha = 45°$ and in figure 6(*b*), h_s is plotted versus the inclination of the weir. It is interesting to note that this curve goes through a minimum for α close to 30°.

5. Waterfalls without a vertical wall

Before looking at the general case of supercritical flows, we consider the special case when $w = 0$ or $t_c = 1$. Such waterfalls have been studied by a number of researchers; see for example Clarke (1965), Chow & Han (1979), Smith & Abd-el-Malek (1983), Goh & Tuck (1985). Here, we look for the range of Froude numbers for which a wave-free solution is possible. We write the hodograph variable ζ as

$$\zeta = (-\ln 2c)^{-\frac{1}{3}}[-\ln c(1+t^2)]^{\frac{1}{3}}\left[1+(1+t)^{2\lambda/\pi}\sum_{m=0}^{\infty} a_m t^m\right]. \tag{13}$$

Note that $\zeta(-1) = 1$ so that the velocity is 1 far upstream. Since the flow considered here is supercritical upstream, it is characterized by the presence of exponentially decreasing terms at $-\infty$. This can be shown by linearizing the problem far upstream about the uniform flow solution. The hodograph variable has the form $\zeta \sim 1 + a_0 e^{\lambda f}$ as $\phi \to -\infty$ where λ is related to the Froude number by the relationship

$$F^2 = \frac{\tan\lambda}{\lambda}. \tag{14}$$

The infinite series is truncated after n terms and n collocation points are chosen on the free surfaces.

Solutions were found for any value of the Froude number greater than 1. Good agreement was found with previous solutions and experimental data (see for example figure 6 in Smith & Abd-el-Malek 1983). Figures 7 and 8 show profiles of the free surfaces for $F = 1$ and $F = 2$. Since the program converged for values of the Froude number as close to $F = 1$ as desired, it was natural to ask the question whether or not solutions which are free of waves also exist for Froude numbers slightly smaller than 1. In order to be able to consider values of F smaller than 1, the expression for ζ must be modified and the exponential decay must be removed. It leads to the following expression for ζ:

$$\zeta = (-\ln 2c)^{-\frac{1}{3}}[-\ln c(1+t^2)]^{\frac{1}{3}}\left[1+\sum_{m=0}^{\infty} a_m(t^m-(-1)^m)\right]. \tag{15}$$

Again, $\zeta(-1) = 1$. The program seemed to converge for Froude numbers down to 0.7. But, as the number of terms retained in the power series was increased and F decreased, oscillations of larger and larger amplitude started to appear on the free surfaces. They may support the conjecture that wave-free waterfalls do not exist for Froude numbers smaller than 1. Chow & Han (1979) have computed a solution for a Froude number of 0.9, but admitted that the existence of subcritical waterfalls was doubtful in steady inviscid flow. Smith & Abd-el-Malek (1983) have computed a solution for a Froude number of 0.8 but no comments were made about the validity of such a solution.

As a simple extension, we have also used the series truncation method to compute 'waterfall-with-lid' flows, as studied by Goh & Tuck (1985) using integral equation methods. That is, a waterfall can be created by allowing a stream to emerge from a channel between rigid horizontal walls. If the upper wall ends sufficiently far upstream relative to the end of the lower wall, the effect of the upper wall is negligible, and Goh & Tuck show results for this limit in agreement with those of Smith & Abd-el-Malek (1983), and hence with our figure 7.

However, the results of Goh & Tuck when the upper wall is more prominent seem to be incorrect. In particular, Goh & Tuck (1985, figure 6) display minimum-speed flows, namely those where there is a stagnation point at the upper wall's end point, and our present results do not support this figure quantitatively. For example, we find that the minimum Froude number for a waterfall from a 'flush' opening where the upper and lower walls end together is $F = 0.703$, not the value $F = 0.551$ quoted by Goh & Tuck (1985). Our corrected value is supported by a separate re-computation using an integral equation method (Tuck 1987) similar to that of Goh & Tuck (1985).

6. Supercritical solutions

In this section, the Froude number F is assumed to be larger than 1. To our knowledge, supercritical solutions with a vertical wall have not been computed in the past. There is now a two-parameter family of solutions. Both the Froude number and the height w of the wall (or equivalently t_c) must be specified. In the previous section, we computed the family of solutions corresponding to $w = 0$.

The expression for the hodograph variable is the same as in the previous section, but the singularity at the corner C and the exact solution without gravity are included (see §7), so that

$$\zeta = -\mathrm{i}\left(\frac{t-t_c}{1-tt_c}\right)^{\frac{1}{2}}(-\ln 2c)^{-\frac{1}{3}}[-\ln c(1+t^2)]^{\frac{1}{3}}\left[1+(1+t)^{2\lambda/\pi}\sum_{m=0}^{\infty} a_m t^m\right]. \tag{16}$$

The numerical procedure is the same as the one described in the previous section. One difference however is that the n unknowns are the n first coefficients $a_0, a_1, \ldots, a_{n-1}$ in the series since t_c is no longer an unknown, but rather is the second parameter.

In order to show on one graph all the solutions for different values of the Froude number F and t_c, we decided to choose y^* as the physical quantity of interest. Recall

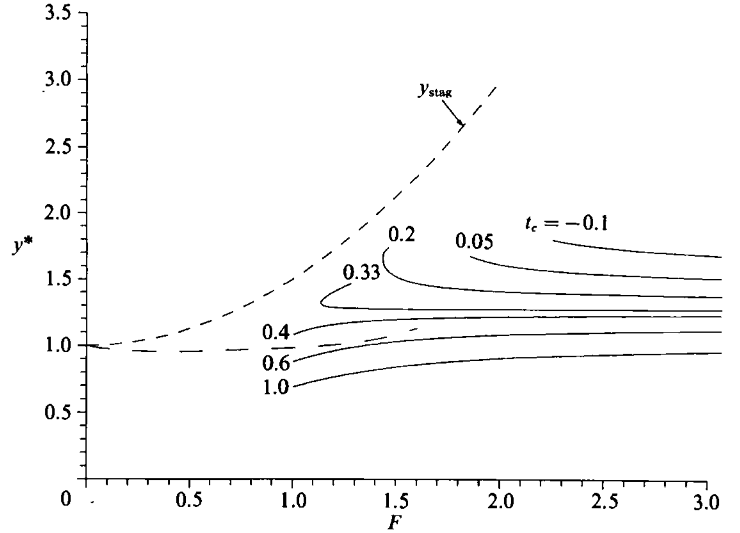

FIGURE 9. Plot of y^* versus F for different values of t_c. y^* is the elevation of the point on the upper free surface which is on the same equipotential as the detachment point S. The short-dashed curve represents the maximum elevation y_{stag} that the free surface can ever reach (see (5)). The long-dashed curve represents the one-parameter family of subcritical weir-flow solutions. It has been continued for Froude numbers between 1 and 1.6 but does not have any particular physical meaning.

that y^* is the elevation of the point S^* on the upper free surface which is on the same equipotential as the detachment point S. Another physical quantity that makes more sense from a physical point of view is the maximum elevation y_{max} of the upper free surface. But for all the solutions for which the upper free surface only goes down as x increases, y_{max} is identically 1. As a trade-off between generality and physical insight, we decided to represent the solutions on two graphs: figure 9 shows a plot of y^* versus F for different values of t_c while figure 10 shows a partial plot of y_{max} for certain values of t_c.

There are two types of curves t_c = constant. The lower curves ($t_c > 0.375$) stop at $F = 1$. These curves could be continued for $F < 1$, but waves would appear on the free surfaces. Along the upper curves ($t_c < 0.375$), there are no solutions below a minimum Froude number F_{min}. The curves turn around at $F = F_{min}$ and start going up towards the curve corresponding to the stagnation level. We conjecture that there is a limiting configuration with a stagnation point on the upper free surface for each value of $t_c < 0.375$. We were able to compute accurately configurations close to the configuration with a stagnation point (see figures 11 and 12). But we were not able to compute the configurations with a stagnation point because the location of the stagnation point is unknown in the [t]-plane and must be found as part of the solution. Moreover, since we chose the collocation points to be equally spaced along the unit circle, it was difficult to obtain an accurate description of the free surface near the stagnation point. If we denote by F_{max} the Froude number corresponding to the solution with a stagnation point (conjecture), we see that, for values of F between F_{min} and F_{max}, there are two solutions to the problem. This phenomenon has already been encountered by Vanden-Broeck & Keller (1987*a*) and by Dias & Vanden-Broeck (1989). It can be viewed as a perturbed bifurcation. One can think of the solutions on the upper part of the curve as 'solitary-wave-type' solutions and of the solutions on the lower part of the curve as 'waterfall-type' solutions.

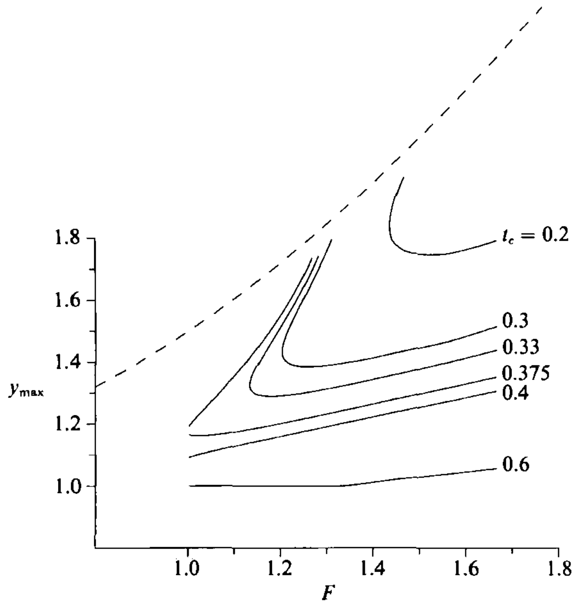

FIGURE 10. Plot of y_{max} versus F for certain values of t_c. y_{max} is the elevation of the highest point on the upper free surface. The short-dashed curve represents the maximum elevation y_{stag} that the free surface can ever reach (see (5)).

FIGURE 11. Computed supercritical flow of 'solitary-wave-type' for $F = 1.27$ and $t_c = 0.375$. The corresponding value of w is 0.17. The horizontal scale is the same as the vertical scale.

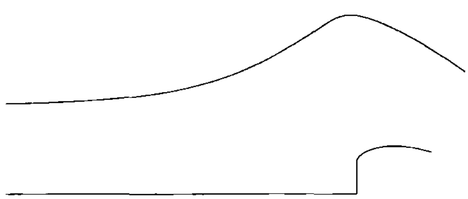

FIGURE 12. Computed supercritical flow of 'solitary-wave-type' for $F = 1.46$ and $t_c = 0.2$. The corresponding value of w is 0.37.

In figures 11 and 12, we show two solitary-wave-type solutions close to the limiting configuration with a stagnation point. In figure 11, the parameters are $F = 1.27$ and $t_c = 0.375$. The resulting w is 0.17 and $y_{max} = 1.73$. The stagnation level corresponding to $F = 1.27$ is 1.81. In figure 12, $F = 1.46$ and $t_c = 0.2$. The resulting w is 0.37 and $y_{max} = 2.00$. The stagnation level corresponding to $F = 1.46$ is 2.07. It is interesting to note that the bump of the solitary wave moves to the left of the wall as the height of the wall decreases. Of course, if the wall is high, we expect the bump of the solitary wave to be far to the right of the wall.

In figures 13 and 14, we show two waterfall-type solutions. In figure 13, the

FIGURE 13. Computed supercritical flow of 'waterfall-type' for $F = 1.66$ and $t_c = 0.4$. The corresponding value of w is 0.16.

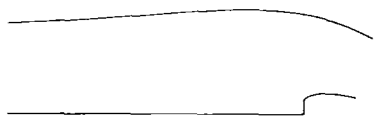

FIGURE 14. Computed supercritical flow of 'waterfall-type' for $F = 1.00$ and $t_c = 0.37$. The corresponding value of w is 0.15.

parameters are $F = 1.66$ and $t_c = 0.4$. The resulting w is 0.16 and $y_{\max} = 1.31$. In figure 14, $F = 1.00$ and $t_c = 0.37$. The resulting w is 0.15 and $y_{\max} = 1.16$.

Except for small values of w, the upper free surface always goes up before falling down. It is easy to generalize the expression (16) for ζ in order to allow the wall CS to be inclined to the vertical. The motivation was to try to obtain solutions in which the upper free surface curves back around to the left before falling under the effect of gravity. In agreement with Goh (1986), we found that the flow almost always 'wants' to go from left to right without curving back. The series truncation method that we used was found not to be appropriate to handle cases where the flow has a tendency to curve back. In a subsequent paper, however, Dias & Christodoulides (1991) show that solutions where the detachment point S is also a stagnation point exist for Froude numbers larger than 2.96. In such solutions, the rising jet does curve around and fall back upon the oncoming stream.

7. Flow in the absence of gravity

When gravity is not present, an exact solution to the problem can be obtained. It is easy to show that

$$\zeta = -\mathrm{i}\left(\frac{t-t_c}{1-tt_c}\right)^{\frac{1}{2}}, \tag{17}$$

and that, consequently,

$$z = \frac{2\mathrm{i}}{\pi}\int_{t_c}^{t}\left(\frac{t-t_c}{1-tt_c}\right)^{-\frac{1}{2}}\frac{1-t}{(1+t)(1+t^2)}\mathrm{d}t. \tag{18}$$

This integral can be integrated exactly to give z as a function of t by using the change of variable

$$t = \frac{t'^2+4t_c}{4+t'^2 t_c}.$$

Note that quite similar solutions were computed by Elcrat & Trefethen (1986) in their computation of flows past a polygonal obstacle. The difference is that they were considering only one free surface, the fluid being of infinite extent. It is easy to show that the angle θ between the jet and the horizontal is given by

$$\theta = \tfrac{1}{4}\pi - \tan^{-1}(t_c). \tag{19}$$

In figure 15, a plot of θ versus the height of the wall w is shown.

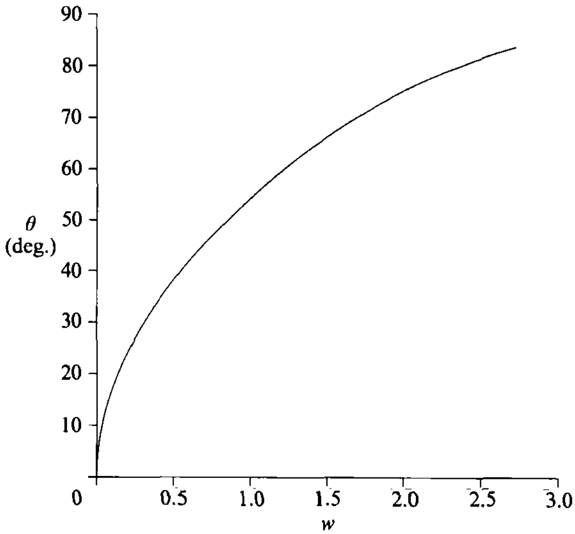

FIGURE 15. Zero-gravity solutions. Plot of θ, the angle between the jet and the horizontal, versus w, the dimensionless height of the wall.

8. Conclusion

Some of the questions asked in the introduction can now be answered:

(i) Thin-weir solutions exist for all values of F between 0 and 1.

(ii) Wave-free waterfalls exist only for $F > 1$. The supercritical solutions are a two-parameter family of solutions, the parameters being F and W/H. For some values of these two parameters, there is non-uniqueness of solution.

(iii) Numerical computations seem to indicate that limiting configurations with a stagnation point on the upper free surface exist. But such solutions could not be computed with the series truncation method used in this paper.

(iv) Unless the detachment point is also a stagnation point, solutions in which the jet curves around and falls back upon the oncoming stream could not be computed.

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